

Bounds on the rate of convergence for inhomogeneous $M/M/S$ systems with either state-dependent transitions, or batch arrivals and service, or both

Alexander Zeifman Anna Korotysheva Yacov Satin
Rostislav Razumchik Victor Korolev Ksenia Kiseleva

Abstract. In this paper one presents method for the computation of convergence bounds for four classes of multiserver queueing systems, described by inhomogeneous Markov chains. Specifically one considers inhomogeneous $M/M/S$ queueing system with possibly state-dependent arrival and service intensities and additionally possible batch arrivals and batch service. The unified approach based on logarithmic norm of linear operators for obtaining sharp upper and lower bounds on the rate of convergence and corresponding sharp perturbation bounds is described. As a side result, one shows by virtue of numerical examples that the approach based on logarithmic norm can also be used for approximation of limiting characteristics (idle probability and mean number of customers in the system) of the considered systems with given approximation error. Extensive numerical examples are provided.

1 Introduction

In this paper one considers the class of Markov processes, which is usually used to describe the evolution of the total number of customers in inhomogeneous Markov queueing systems. Suppose that the system's state space is $\mathcal{X} = \{0, 1, 2, \dots\}$, where the (i) -th state means that there are i customers in the system. Throughout the paper it is assumed that all possible transition intensities between states are non-random functions of time and may depend on the state of the system state.

There are two most common problems related to such systems: the computation of time-dependent distribution of the state probabilities and the limiting distribution (for example, in case of periodic intensities); computation of the rate of convergence and perturbation bounds.

This paper deals with the second problem related to the following classes of inhomogeneous Markov queueing systems with arbitrary finite number of servers S :

- I. inhomogeneous $M/M/S$ queueing system with possibly state-dependent arrival and service intensities;
- II. inhomogeneous $M/M/S$ queueing system with state-independent batch arrivals and state-dependent service intensity;
- III. inhomogeneous $M/M/S$ queueing system with state-dependent arrival intensity and batch service;
- IV. inhomogeneous $M/M/S$ queueing system with state-independent batch arrivals and batch service.

Here for these four system classes one describes the unified approach based on logarithmic norm of linear operators for obtaining sharp upper and lower bounds on the rate of convergence and corresponding sharp perturbation bounds.

This unified approach has already been successfully applied to system from the Ist and IVth class. Specifically for the inhomogeneous $M/M/1$ system with state-dependent arrival and service intensities, as well as for the state-independent inhomogeneous $M/M/S$ system the bounds were firstly obtained in [24], [26] and [4]. Systems belonging to the IVth class have been studied in a number of papers (see, for example, [20], [12], [14]) and the results related to convergence have been also obtained in [21, 30]. Here one demonstrates that the approach is also suitable for the systems from the IInd and III^d class and thus offers a unified way toward analysis of ergodicity properties of such Markov chains.

The approach is based on the special properties of linear systems of differential equations with non-diagonally non-negative matrices. Specifically, if the column-wise sums of the elements of this matrix are identical and equal to, say, $-\alpha^*(t)$, then the exact upper bound of order $\exp \left\{ - \int_0^t \alpha^*(u) du \right\}$ can be obtained for the rate of convergence of the solutions of the system in

the corresponding metric. Moreover, if the column-wise sums of the absolute values of the elements of this matrix are identical and equal to, say, $\chi^*(t)$, then the exact lower bound of order $\exp \left\{ - \int_0^t \chi^*(u) du \right\}$ can be obtained for the convergence rate as well. The bounds are obtained in three steps. At first step one excludes the (0) state from the forward Kolmogorov system of differential equations and thus obtains the new system with the new intensity matrix which is, in general, not non-diagonally non-negative. The second step is to transform the new intensity matrix in such a way that non-diagonally elements are non-negative and which leads to (loosely speaking) least distance between specifically defined upper and lower bounds. At third step one uses the logarithmic norm for the estimation of the convergence rate.

Here the key step is the second one. The transformation is made using a sequence of positive numbers $\{d_i, i \geq 1\}$, which does not have any probabilistic sense and can be considered as an analogue of Lyapunov functions. For the detailed discussion on application of logarithmic norm and related techniques one can refer to the series of papers [4, 8, 26, 27, 30, 34].

The advantages of this three-step approach is that it allows one to deal with time-homogeneous and time-inhomogeneous processes and it leads to exact both upper and lower bounds for the convergence rate. In time-homogeneous case (of the four classes of systems introduced above) the approach allows one to obtain the correspondent bounds for the decay parameter and gives an explicit bounds in total variation norm (see *Theorem 2*).

The proposed approach allows one also to address the problem of computation of the limiting distribution of the inhomogeneous Markov chain from a different perspective. In general there are several approaches, which allow one to obtain more or less accurate solutions. These are the exact and approximate numerical solution of the system of differential equations, approaches assuming piecewise constant parameters and approaches based on modified system characteristics. For the review of many results one can refer to [16]. Although using the proposed approach one cannot determine the state probabilities as functions of time t , it is possible to compute approximately the limiting distribution, while having analytically computable expressions for the approximation errors. Using truncation techniques, which were developed in [27, 31], in the numerical one presents the results of the computation of the limiting characteristics in inhomogeneous $M/M/S$ systems of each of the four classes described above. The most interesting insight from the experiments is the following. Choose the arrival and service intensi-

ties in inhomogeneous $M/M/S$ (I^{st} class). Then if one uses these intensities (after a certain modification allowing bulk arrivals and group services) in inhomogeneous $M/M/S$ from the II^{nd} , III^d or IV^{th} class, then the limiting mean number of customers for both systems coincide, while idle probabilities do not.

The paper is structured as follows. In the next section one gives the general description of the system under consideration and introduces the necessary notation. Section 3 contains the main result of the paper i.e. the theorem which specifies the convergence bounds. Section 4 provides explicit expressions for functions needed to compute convergence bounds for four special cases of the considered system. In the last two sections one provides extensive numerical examples and gives directions of further research.

2 System description and definitions

Let the integer-valued time-dependent random variable $X(t)$ denote the total number of customers at time t in a markovian queueing system. Then the process $\{X(t), t \geq 0\}$ is a (possibly inhomogeneous) continuous-time Markov chain with state space $\mathcal{X} = \{0, 1, 2, \dots\}$. Denote by $p_{ij}(s, t) = P\{X(t) = j | X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that the Markov chain $X(t)$ is in state i at time t . Let $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$ be probability distribution vector at instant t . Throughout the paper we assume that in an element of time h the possible transitions and their associated probabilities are

$$p_{ij}(t, t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t, h), & \text{if } j \neq i \\ 1 + q_{ii}(t)h + \alpha_i(t, h), & \text{if } j = i, \end{cases} \quad i, j \in \mathcal{X}, \quad (1)$$

where all $\alpha_i(t, h)$ are $o(h)$ uniformly in i , i.e. $\sup_i |\alpha_i(t, h)| = o(h)$ and

$$q_{ii}(t) = - \sum_{k \in \mathcal{X}, k \neq i} q_{ik}(t).$$

Applying the standard approach developed in [8, 26, 27] it is assumed that all the intensity functions $q_{ij}(t)$ are linear combinations of a finite number of locally integrable on $[0, \infty)$ non-negative functions.

The matrix $Q(t) = (q_{ij}(t))_{i,j=0}^{\infty}$ is the intensity matrix of the chain $\{X(t), t \geq 0\}$. Henceforth it is assumed that the $Q(t)$ is essentially bounded,

i. e.

$$\sup_i |q_{ii}(t)| = L(t) \leq L < \infty, \quad (2)$$

for almost all $t \geq 0$.

Probabilistic dynamics of the process $\{X(t), t \geq 0\}$ is given by the forward Kolmogorov system

$$\frac{d}{dt} \mathbf{p}(t) = A(t) \mathbf{p}(t), \quad (3)$$

where $A(t) = Q^T(t)$ is the transposed intensity matrix.

Throughout the paper by $\|\cdot\|$ we denote the l_1 -norm, i. e. $\|\mathbf{p}(t)\| = \sum_{k \in \mathcal{X}} |p_k(t)|$, and $\|Q(t)\| = \sup_{j \in \mathcal{X}} \sum_{i \in \mathcal{X}} |q_{ij}|$. Let Ω be a set all stochastic vectors, i. e. l_1 vectors with non-negative coordinates and unit norm. Hence we have $\|A(t)\| = 2 \sup_{k \in \mathcal{X}} |q_{kk}(t)| \leq 2L$ for almost all $t \geq 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0; \infty)$. Therefore we can consider (3) as a differential equation in the space l_1 with bounded operator.

It is well known (see [1]) that the Cauchy problem for differential equation (3) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

Denote by $E(t, k) = E(X(t) | X(0) = k)$ the conditional expected number of customers in the system at instant t , provided that initially (at instant $t = 0$) k customers were present in the system. Then $E_p(t) = \sum_{k \geq 0} E(t, k) p_k(0)$ is the unconditional expected number of customers in the system at instant t , given that the initial distribution of the total number of customers was $\mathbf{p}(0)$.

In order to obtain perturbation bounds we consider a class of perturbed Markov chains $\{\bar{X}(t), t \geq 0\}$ defined on the same state space \mathcal{X} as the original Markov chain $\{X(t), t \geq 0\}$, with the intensity matrix $\bar{A}(t)$ and the same restrictions as imposed on $A(t)$. It is assumed that $\|\hat{A}(t)\| = \|A(t) - \bar{A}(t)\| \leq \varepsilon$, for almost all $t \geq 0$, which means the perturbations are considered to be small.

Before proceeding to the derivation of the main results of the paper, we recall two definitions. Recall that a Markov chain $\{X(t), t \geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \rightarrow 0$ as $t \rightarrow \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (3). A Markov chain $\{X(t), t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t \rightarrow \infty} (\varphi(t) - E(t, k)) = 0$ for any k .

3 Main results

Recall that one has introduced $A(t)$ as the transposed intensity matrix $Q(t)$. Thus it has the form

$$A(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & \cdots & a_{0r}(t) & \cdots \\ a_{10}(t) & a_{11}(t) & \cdots & a_{1r}(t) & \cdots \\ a_{20}(t) & a_{21}(t) & \cdots & a_{2r}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r0}(t) & a_{r1}(t) & \cdots & a_{rr}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (4)$$

where $a_{ii}(t) = -\sum_{k \in \mathcal{X}, k \neq i} a_{ki}(t)$. Since $p_0(t) = 1 - \sum_{i=1}^{\infty} p_i(t)$ due to normalization condition, one can rewrite the system (3) as follows:

$$\frac{d}{dt} \mathbf{z}(t) = B(t) \mathbf{z}(t) + \mathbf{f}(t), \quad (5)$$

where

$$\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \dots)^T, \quad \mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T, \\ B = (b_{ij}(t))_{i,j=1}^{\infty} = \begin{pmatrix} a_{11}(t) - a_{10}(t) & a_{12}(t) - a_{10}(t) & \cdots & a_{1r}(t) - a_{10}(t) & \cdots \\ a_{21}(t) - a_{20}(t) & a_{22}(t) - a_{20}(t) & \cdots & a_{2r}(t) - a_{20}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1}(t) - a_{r0}(t) & a_{r2}(t) - a_{r0}(t) & \cdots & a_{rr}(t) - a_{r0}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (6)$$

See detailed discussion of this transformation in [8, 26, 27]. Let $\{d_i, i \geq 1\}$ with $d_1 = 1$ be an increasing sequence of positive numbers. Put

$$W = \inf_{i \geq 1} \frac{d_i}{i}. \quad (7)$$

and denote by D the upper triangular matrix of the following form:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (8)$$

Let l_{1D} be the corresponding space of sequences

$$l_{1D} = \{\mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T \mid \|\mathbf{z}(t)\|_{1D} \equiv \|D\mathbf{z}(t)\|_1 < \infty\}$$

and introduce also the auxiliary norm $\|\cdot\|_{1E}$ defined as $\|\mathbf{z}(t)\|_{1E} = \sum_{k=1}^{\infty} k|p_k(t)|$. Then in $\|\cdot\|_{1D}$ norm the following two inequalities hold:

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1D} &= d_1 \left| \sum_{i=1}^{\infty} p_i(t) \right| + d_2 \left| \sum_{i=2}^{\infty} p_i(t) \right| + d_3 \left| \sum_{i=3}^{\infty} p_i(t) \right| + \dots \geq \\
&\geq \left(\left| \sum_{i=1}^{\infty} p_i(t) \right| + \left| \sum_{i=2}^{\infty} p_i(t) \right| + \left| \sum_{i=3}^{\infty} p_i(t) \right| + \dots \right) \geq \\
&\geq \frac{1}{2} \left(\left(\left| \sum_{i=1}^{\infty} p_i(t) \right| + \left| \sum_{i=2}^{\infty} p_i(t) \right| \right) + \left(\left| \sum_{i=2}^{\infty} p_i(t) \right| + \left| \sum_{i=3}^{\infty} p_i(t) \right| \right) + \dots \right) \geq \\
&\geq \frac{1}{2} \sum_{i=1}^{\infty} |p_i(t)| = \frac{1}{2} \|\mathbf{z}(t)\|_1, \quad (9)
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1E} &= \sum_{k=1}^{\infty} k|p_k(t)| = \sum_{k=1}^{\infty} \frac{k}{d_k} d_k |p_k(t)| \leq W^{-1} \sum_{k=1}^{\infty} d_k |p_k(t)| = \\
&= W^{-1} \sum_{k=1}^{\infty} d_k \left| \sum_{i=k}^{\infty} p_i(t) - \sum_{i=k-1}^{\infty} p_i(t) \right| \leq W^{-1} \sum_{k=1}^{\infty} d_k \left(\left| \sum_{i=k}^{\infty} p_i(t) \right| + \left| \sum_{i=k-1}^{\infty} p_i(t) \right| \right) \leq \\
&\leq \frac{2}{W} \sum_{k=1}^{\infty} d_k \left| \sum_{i=k}^{\infty} p_i(t) \right| \leq \frac{2}{W} \|\mathbf{z}(t)\|_{1D}. \quad (10)
\end{aligned}$$

Consider the equation (5) in the space l_{1D} , where $B(t)$ and $\mathbf{f}(t)$ are locally integrable on $[0, +\infty)$. Let one compute the logarithmic norm of operator function $B(t)$. The motivation behind this can be found in [4] and detailed proofs are provided in [25]. Recall that the logarithmic norm of operator function $B(t)$ is defined as

$$\gamma(B(t)) = \lim_{h \rightarrow +0} h^{-1} (\|I + hB(t)\| - 1).$$

Denote by $V(t, s) = V(t)V^{-1}(s)$ the Cauchy operator of the equation (5). Then the important inequality holds

$$e^{-\int_s^t \gamma(-B(u)) du} \leq \|V(t, s)\| \leq e^{\int_s^t \gamma(B(u)) du}.$$

Further, for an operator function from l_1 to itself one has the simple formula

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right).$$

Moreover for the logarithmic norm of the operator function $B(t)$ in $\|\cdot\|_{1D}$ norm the following equality holds:

$$\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1})_1.$$

Denote the elements of the matrix $DB(t)D^{-1}$ by $b_{ij}^*(t)$ i.e. $DB(t)D^{-1} = (b_{ij}^*(t))_{i,j=1}^\infty$. Assume that

$$b_{ij}^*(t) \geq 0, \quad i \neq j, \quad t \geq 0. \quad (11)$$

Put

$$\alpha_i(t) = \sum_{j=0}^\infty b_{ji}^*(t), \quad \chi_i(t) = -\sum_{j=0}^\infty |b_{ji}^*(t)|, \quad i \geq 1, \quad (12)$$

and let $\alpha(t)$ and $\beta(t)$ denote the least lower and the least upper bound of the sequence of functions $\{\alpha_i(t), i \geq 1\}$ and χ denote the least upper bound of $\{\chi_i(t), i \geq 1\}$ i.e.

$$\alpha(t) = \inf_{i \geq 1} \alpha_i(t), \quad \beta(t) = \sup_{i \geq 1} \alpha_i(t), \quad (13)$$

$$\chi(t) = \sup_{i \geq 1} \chi_i(t). \quad (14)$$

Then the logarithmic norms of $B(t)$ and $(-B(t))$ are equal to

$$\gamma(B(t))_{1D} = \sup_i \alpha_i(t) = -\alpha(t), \quad \gamma(-B(t))_{1D} = \sup_i \chi_i(t) = \chi(t).$$

If now one defines $\mathbf{v}(t) = D(\mathbf{p}^*(t) - \mathbf{p}^{**}(t))$, then the following equation holds

$$\frac{d}{dt} \mathbf{v}(t) = DB(t)D^{-1} \mathbf{v}(t), \quad (15)$$

Notice that due to (11), the inequality $\mathbf{v}(s) \geq \mathbf{0}$ implies that $\mathbf{v}(t) \geq \mathbf{0}$ for any $t \geq s$. Hence

$$\frac{d}{dt} \sum_{i=1}^\infty v_i(t) \geq -\beta(t) \sum_{i=1}^\infty v_i(t), \quad (16)$$

and one can obtain establish the following theorem.

Theorem 1. *Let there exist an increasing sequence $\{d_j, j \geq 1\}$ of positive numbers with $d_1 = 1$, such that (11) holds, and $\alpha(t)$ defined by (13) satisfies*

$$\int_0^\infty \alpha(t) dt = +\infty. \quad (17)$$

Then the Markov chain $\{X(t), t \geq 0\}$ is weakly ergodic and the following bounds hold:

$$e^{-\int_s^t \chi(u) du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D} \leq \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq e^{-\int_s^t \alpha(u) du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \quad (18)$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\int_s^t \alpha(u) du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \quad (19)$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \leq \frac{2}{W} e^{-\int_s^t \alpha(u) du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \quad (20)$$

for any initial conditions $s \geq 0$, $\mathbf{p}^(s)$, $\mathbf{p}^{**}(s)$ and any $t \geq s$.*

If in addition $D(\mathbf{p}^(s) - \mathbf{p}^{**}(s)) \geq \mathbf{0}$, then*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \geq e^{-\int_s^t \beta(u) du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \quad (21)$$

for any $0 \leq s \leq t$.

One can also obtain the corresponding perturbation bounds. For the first results in this direction see [9, 10, 11, 22], for the stronger results see [19] and for the general approach see [32]. The respective uniform in time truncation bounds can be obtained via techniques proposed in [31, 35].

If the Markov chain is homogeneous, then all elements $b_{ij}^*(t)$ of the matrix $DB(t)D^{-1}$ do not depend on t i.e. the quantities in (13) are constants. Thus instead of general bounds given by Theorem 1, one can specify then and obtain the following theorem.

Theorem 2. *Let there exist an increasing sequence $\{d_j, j \geq 1\}$ of positive numbers with $d_1 = 1$, such that (11) holds, and $\alpha(t) = \alpha$ defined by (13) is positive i.e. $\alpha > 0$. Then the Markov chain $\{X(t), t \geq 0\}$ is strongly ergodic and the following bounds hold:*

$$e^{-\chi t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D} \leq \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq e^{-\alpha t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \quad (22)$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\alpha t} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \quad (23)$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \leq \frac{2}{W} e^{-\alpha t} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \quad (24)$$

for any initial conditions $s \geq 0$, $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$ and any $t \geq 0$. If in addition $D(\mathbf{p}^*(0) - \mathbf{p}^{**}(0)) \geq \mathbf{0}$, then

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \geq e^{-\beta t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \quad (25)$$

for any $t \geq 0$.

For the decay parameter α^* defined as

$$\lim_{t \rightarrow \infty} (p_{ij}(t) - \pi_j) = O(e^{-\alpha^* t}),$$

where $\{\pi_j, j \geq 0\}$ are the stationary probabilities of the chain, it holds that $\alpha^* \geq \alpha$.

Notice that some additional results related to *Theorem 2* can also be found in [4, 7, 24]. If one assumes that the intensities $q_{ij}(t)$ are 1-periodic in t i.e. $q_{ij}(t)$ are periodic functions and the length of the period is equal to one, then the Markov chain $\{X(t), t \geq 0\}$ has the limiting 1-periodic limiting regime. Under the assumptions of *Theorem 1* the Markov chain $\{X(t), t \geq 0\}$ is exponentially weakly ergodic. The detailed discussion of this results is given in [27].

Till the end of this section one presents a bit more detailed analysis of two special cases: homogeneous case and the case with periodic intensities. Firstly note that in the both cases there exist positive R and a such that

$$e^{-\int_s^t \alpha(u) du} \leq R e^{-a(t-s)} \quad (26)$$

for any $0 \leq s \leq t$. Hence the Markov chain $\{X(t), t \geq 0\}$ is *exponentially* weakly ergodic. Indeed, if the Markov chain $\{X(t), t \geq 0\}$ is homogeneous, then one may put $R = 1$, $a = \alpha$ given by (13). If all the intensity functions $q_{ij}(t)$ are 1-periodic in t , then one may put

$$a = \int_0^1 \alpha(t) dt, \quad R = e^K, \quad K = \sup_{|t-s| \leq 1} \int_s^t \alpha(u) du.$$

By doing so, for any solution of (5) the following bound holds:

$$\begin{aligned} & \|\mathbf{z}(t)\|_{1D} \leq \\ & \|V(t)\|_{1D} \|\mathbf{z}(0)\|_{1D} + \int_0^t \|V(t, \tau)\|_{1D} \|\mathbf{f}(\tau)\|_{1D} d\tau \leq \\ & R e^{-at} \|\mathbf{z}(0)\|_{1D} + \frac{FR}{a}, \end{aligned} \quad (27)$$

where F is such that $\|\mathbf{f}(t)\|_{1D} \leq F$ for almost all $t \in [0, 1]$. Hence one has the upper bound for the limit

$$\limsup_{t \rightarrow \infty} \|\mathbf{z}(t)\|_{1D} \leq \frac{FR}{a}, \quad (28)$$

for any initial condition and

$$\|\mathbf{p}(0) - \mathbf{e}_0\|_{1D} = \|\mathbf{p}(0)\|_{1D} = \|\mathbf{z}(0)\|_{1D} \leq \limsup_{t \rightarrow \infty} \|\mathbf{z}(t)\|_{1D}, \quad (29)$$

where \mathbf{e}_i denotes the unit vector of zeros with 1 in the i -th place. If the initial distribution is $\mathbf{p}^{**}(0) = \mathbf{e}_0$ then $\mathbf{z}^{**}(0) = \mathbf{0}$, $\mathbf{z}(t) \geq 0$ for any $\mathbf{p}^*(0)$ and any $t \geq 0$. Therefore

$$\begin{aligned} \|\mathbf{z}(t)\|_{1D} &= d_1 p_1 + (d_1 + d_2) p_2 + (d_1 + d_2 + d_3) p_3 + \dots = \\ &= d_1 p_1 + \frac{d_1 + d_2}{2} 2p_2 + \frac{d_1 + d_2 + d_3}{3} 3p_3 + \dots \geq \inf_k \frac{d_1 + \dots + d_k}{k} \|\mathbf{z}(t)\|_{1E}, \end{aligned}$$

and one can use $W^* = \inf_k \frac{d_1 + \dots + d_k}{k}$ instead of $W = \inf_k \frac{d_k}{k}$, given by (7) in all the bounds on the rate of convergence. Finally, for the considered two special cases one has the following two corollaries.

Corollary 1. *Let $\{X(t), t \geq 0\}$ be a homogeneous Markov chain and let there exist an increasing sequence $\{d_j, j \geq 1\}$ of positive numbers with $d_1 = 1$ such that (11) holds and in addition $\alpha > 0$. Then the Markov chain $\{X(t), t \geq 0\}$ is exponentially ergodic and the following bounds hold:*

$$\|\pi - \mathbf{p}(t, 0)\| \leq \frac{4F}{\alpha} e^{-\alpha t}, \quad (30)$$

$$|\phi - E(t, 0)| \leq \frac{F}{\alpha W^*} e^{-\alpha t}, \quad (31)$$

where $\pi = (\pi_0, \pi_1, \dots)^T$ denotes the vector of stationary probabilities of the chain and $\phi = \sum_{j=0}^{\infty} j \pi_j$ and $\mathbf{p}(0, 0) = \mathbf{e}_0$.

Corollary 2. *Assume all intensity function of the Markov chain $\{X(t), t \geq 0\}$ be 1-periodic in t . Let there exist an increasing sequence $\{d_j, j \geq 1\}$ of positive numbers with $d_1 = 1$ such that (11) holds and in addition $\int_0^1 \alpha(t) dt = a > 0$. Then the Markov chain $\{X(t), t \geq 0\}$ is exponentially weakly ergodic and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}(t, 0)\| \leq \frac{4FR}{a} e^{-at}, \quad (32)$$

$$|\phi(t) - E(t, 0)| \leq \frac{FR}{aW_*} e^{-at}, \quad (33)$$

where $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$ denotes the vector of limiting probabilities of the chain and $\phi(t) = \sum_{j=0}^{\infty} j\pi_j(t)$ and $\mathbf{p}(0, 0) = \mathbf{e}_0$.

If the state space of the Markov chain is finite there exist a number of special results (one can refer to [4, 7, 29]).

4 Convergence bounds

In order to apply the results of the *Theorem 1* and *Theorem 2* and to obtain the convergence bounds for the system from either $I^{st} - IV^{th}$ class, one has to know the exact expressions for the functions $\alpha_i(t)$ and $\chi_i(t)$, given by (13) and (14). In this section one provides the expressions for $\alpha_i(t)$ and $\chi_i(t)$ for single server systems from classes $I^{st} - IV^{th}$. Then one shows how these expression change, when one considers to multiple server case.

4.1 Inhomogeneous $M/M/S$ queueing system with batch arrivals and state-dependent service intensity

Consider the queueing system $M/M/1$ queueing system with time-dependent arrival and service intensities. Let $\lambda_k(t)$ be the arrival intensity of the batch, containing k customers, at instant t and $\mu_k(t)$ be the service intensity at instant t if the total number of customers in the system is equal to k . Then the transposed intensity matrix has the form

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & 0 & 0 & 0 & \dots \\ \lambda_1(t) & a_{11}(t) & \mu_2(t) & 0 & 0 & \dots \\ \lambda_2(t) & \lambda_1(t) & a_{22}(t) & \mu_3(t) & 0 & \dots \\ \lambda_3(t) & \lambda_2(t) & \lambda_1(t) & a_{33}(t) & \mu_4(t) & \dots \\ \lambda_4(t) & \lambda_3(t) & \lambda_2(t) & \lambda_1(t) & a_{44}(t) & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (34)$$

where diagonal elements of $A(t)$ are such that all column sums are equal to zero for any $t \geq 0$. As the assumption (11) is fulfilled, it holds that

$$\alpha_j(t) = \mu_j(t) - \frac{d_{j-1}}{d_j} \mu_{j-1}(t) + \sum_{i=1}^{\infty} \left(1 - \frac{d_{i+j}}{d_j}\right) \lambda_i(t), \quad (35)$$

and

$$\chi_j(t) = \mu_j(t) + \frac{d_{j-1}}{d_j} \mu_{j-1}(t) + \sum_{i=1}^{\infty} \left(1 + \frac{d_{i+j}}{d_j}\right) \lambda_i(t). \quad (36)$$

Therefore, the *Theorem 1* and *Theorem 2* hold for these $\alpha_i(t)$ and $\chi_i(t)$.

Consider now the queueing system $M/M/S$ queueing system with $S > 1$ servers, time-dependent arrival and service intensities. Customers arrive at the system in batches of size not greater than S . Assume that the arrival intensity of batch, containing k customers, at instant t is equal to $\lambda_k(t) = \frac{1}{S^k} \lambda(t)$ if $1 \leq k \leq S$ and $\lambda_k(t) = 0$ if $k > S$. Denote the service intensity at instant t by $\mu_k(t)$ and assume that $\mu_k(t) = \min(k, S) \mu(t)$. For the assumed values of $\lambda_k(t)$ and $\mu_k(t)$, and the expressions for $\alpha_i(t)$ and $\chi_i(t)$ given above, the *Theorem 1* and *Theorem 2* hold.

4.2 Inhomogeneous $M/M/S$ queueing system with batch service and state-dependent arrival intensity

Consider the queueing system $M/M/1$ queueing system with time-dependent arrival and service intensities. But now let $\lambda_k(t)$ be the arrival intensity of k customers at instant t if the total number of customers in the system is equal to k and $\mu_k(t)$ be the service intensity at instant t of a group of k customers. Then the transposed intensity matrix has the form

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \mu_4(t) & \mu_5(t) & \cdots \\ \lambda_0(t) & a_{11}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \mu_4(t) & \cdots \\ 0 & \lambda_1(t) & a_{22}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \cdots \\ 0 & 0 & \lambda_2(t) & a_{33}(t) & \mu_1(t) & \mu_2(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (37)$$

where $a_{ii}(t) = -\sum_{k=1}^i \mu_k(t) - \lambda_i(t)$. Then *Theorem 1* and *Theorem 2* hold for

$$\alpha_i(t) = \mu_i(t) - \sum_{k=1}^{i-1} (\mu_{i-k}(t) - \mu_i(t)) \frac{d_k}{d_i} + \lambda_{i-1}(t) - \frac{d_{i+1}}{d_i} \lambda_i(t), \quad (38)$$

$$\chi_i(t) = \mu_i(t) + \sum_{k=1}^{i-1} (\mu_{i-k}(t) - \mu_i(t)) \frac{d_k}{d_i} + \lambda_{i-1}(t) + \frac{d_{i+1}}{d_i} \lambda_i(t). \quad (39)$$

Consider again the queueing system $M/M/S$ queueing system with $S > 1$ servers, time-dependent arrival and service intensities. The customers are served in batches of size not greater than S . Assume that the service intensity is $\mu_k(t) = \frac{1}{k}\mu(t)$ if $1 \leq k \leq S$ and $\mu_k(t) = 0$ if $k > S$. Denote the arrival intensity by $\lambda(t)$. For the assumed values of $\lambda_k(t)$ and $\mu_k(t)$, and the expressions for $\alpha_i(t)$ and $\chi_i(t)$ given above, the *Theorem 1* and *Theorem 2* hold.

4.3 Inhomogeneous $M/M/S$ queueing system with batch arrivals and service

Consider the queueing system $M/M/1$ queueing system with time-dependent arrival and service intensities. Customer arrive and are served in batches. Let $\lambda_k(t)$ and $\mu_k(t)$ be the arrival and service intensity of a group of k customers. This queueing system has been extensively studied with respect to the rate of convergence, truncation and perturbation bounds in [21, 30]. The transposed intensity matrix in this case has the following form:

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \cdots \\ \lambda_1(t) & a_{11}(t) & \mu_1(t) & \mu_2(t) & \cdots \\ \lambda_2(t) & \lambda_1(t) & a_{22}(t) & \mu_1(t) & \cdots \\ \lambda_3(t) & \lambda_2(t) & \lambda_1(t) & a_{33}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (40)$$

and $a_{ii}(t) = -\sum_{k=1}^i \mu_k(t) - \sum_{k=1}^{\infty} \lambda_k(t)$. Therefore, *Theorem 1* and *Theorem 2* hold for

$$\alpha_i(t) = -a_{ii}(t) - \sum_{k=1}^{i-1} (\mu_{i-k}(t) - \mu_i(t)) \frac{d_k}{d_i} - \sum_{k \geq 1} \lambda_k(t) \frac{d_{k+i}}{d_i}, \quad (41)$$

and

$$\chi_i(t) = -a_{ii}(t) + \sum_{k=1}^{i-1} (\mu_{i-k}(t) - \mu_i(t)) \frac{d_k}{d_i} + \sum_{k \geq 1} \lambda_k(t) \frac{d_{k+i}}{d_i}. \quad (42)$$

Consider again the queueing system $M/M/S$ queueing system with $S > 1$ servers, time-dependent arrival and service intensities. Assume the arrivals and service appear in batches of size not greater than S . Assume that the arrival intensity of k customers at instant t is equal to $\lambda_k(t) = \frac{1}{S^k} \lambda(t)$ if

$1 \leq k \leq S$ and $\lambda_k(t) = 0$ if $k > S$; the service intensity is assume to be equal to $\mu_k(t) = \frac{1}{k}\mu(t)$ if $1 \leq k \leq S$ and $\mu_k(t) = 0$ if $k > S$. For the assumed values of $\lambda_k(t)$ and $\mu_k(t)$, and the expressions for $\alpha_i(t)$ and $\chi_i(t)$ given above, the *Theorem 1* and *Theorem 2* hold.

4.4 Inhomogeneous $M/M/S$ queueing system with state-dependent arrival and service intensities

If in the queueing system $M/M/1$ queueing system the arrival $\lambda_n(t)$ and service intensities $\mu_n(t)$ are time-dependent and also depend on the total number of customers n in the system then the queue-length process for a general Markovian queue with intensities $\lambda_n(t)$ is the inhomogeneous birth-death process with birth and death intensities equal to $\lambda_n(t)$ and $\mu_n(t)$. Thus *Theorem 1* and *Theorem 2* hold with

$$\alpha_j(t) = \mu_j(t) - \frac{d_{j-1}}{d_j}\mu_{j-1}(t) + \lambda_{j-1}(t) - \frac{d_{j+1}}{d_j}\lambda_j(t), \quad (43)$$

and

$$\chi_j(t) = \mu_j(t) + \frac{d_{j-1}}{d_j}\mu_{j-1}(t) + \lambda_{j-1}(t) + \frac{d_{j+1}}{d_j}\lambda_j(t). \quad (44)$$

Consider again the ordinary queueing system $M/M/S$ queueing system with $S > 1$ servers, time-dependent arrival and service intensities $\lambda(t)$ and $\mu_n(t) = \min(n, S)\mu(t)$ correspondingly. For the assumed values of $\lambda_k(t)$ and $\mu_k(t)$, and the expressions for $\alpha_i(t)$ and $\chi_i(t)$ given above, the *Theorem 1* and *Theorem 2* hold.

5 Numerical examples

The purpose of the numerical section is two-fold. Firstly one demonstrates that the convergence bounds obtained in the previous section can indeed be computed. Having fixed the arrival and service intensities in the inhomogeneous $M/M/S$ queueing system with state-independent arrival and service intensities (I^{st} class), one specifies the sequence $\{d_i, i \geq 1\}$ and provides corresponding bounds using *Corollary 2*. Secondly one shows that the approach proposed in this paper can be used to compute approximations for the limiting characteristics of the systems with a given approximation error.

The characteristics under consideration are the limiting idle probability and the limiting mean number of customers in the system.

The system considered are:

- (i) inhomogeneous $M/M/100$ queueing system with state-independent arrival and service intensities;
- (ii) inhomogeneous $M/M/100$ queueing system with state-independent batch arrivals;
- (iii) inhomogeneous $M/M/100$ queueing system with state-independent batch service.
- (iv) inhomogeneous $M/M/100$ queueing system with state-independent batch arrivals and batch service.

All transition intensities are assumed to be periodic functions of time. Customer in all three systems are served in FCFS order. The inhomogeneous $M/M/S$ consists of single infinite capacity queue and 100 servers. Arrivals happen according to the inhomogeneous Poisson process with the intensity $\lambda^*(t; i)$ equal to

$$\lambda^*(t; i) = i(1 + \sin 2\pi t), \quad i > 0, \quad t > 0.$$

Whenever the server becomes free, a customer from the queue (if there is any) enters server and get served according to exponential distribution with the intensity

$$\mu^*(t) = 3 + \cos 2\pi t.$$

In the inhomogeneous $M/M/100$ queueing system with batch arrivals it is assumed that customers arrive in batches in accordance with a inhomogeneous Poisson process of intensity $\lambda^*(t; i) / \sum_{i=1}^S (Si)^{-1}$. The size of the arriving group is a random variable with discrete probability distribution $(Sk)^{-1} / \sum_{i=1}^S (Si)^{-1}$, $1 \leq k \leq S$. The sizes and interarrival times of successive arriving groups are stochastically independent. Thus the total arrival intensity is $\lambda_k(t, i) = \frac{1}{Sk} \lambda^*(t, i)$ if $1 \leq k \leq S$ and $\lambda_k(t, i) = 0$ if $k > S$. Whenever the server becomes free, a customer from the queue (if there is any) enters server and get served according to exponential distribution with the intensity $\mu_k(t) = \min(k, S) \mu^*(t)$.

In the inhomogeneous $M/M/100$ queueing system with batch service customers arrive in accordance with a inhomogeneous Poisson process of the same intensity $\lambda^*(t, i)$. But the service happens in batches of size not greater than S and the service times are exponentially distribution with the service intensity equal to $\mu_k(t) = \frac{1}{k} \mu(t)$ if $1 \leq k \leq S$ and $\mu_k(t) = 0$ if $k > S$.

Finally, in the inhomogeneous $M/M/100$ queueing system with batch arrivals and batch service, the total arrival intensity is $\lambda_k(t, i) = \frac{1}{S_k} \lambda^*(t, i)$ if $1 \leq k \leq S$ and $\lambda_k(t, i) = 0$ if $k > S$ and the service intensity is $\mu_k(t) = \frac{1}{k} \mu(t)$ if $1 \leq k \leq S$ and $\mu_k(t) = 0$ if $k > S$.

Let us find the convergence bounds in case (i) i.e. for the inhomogeneous $M/M/100$ queueing system with state-independent arrival and service intensities. Let $i = 50$ i.e. arrival intensity is equal to

$$\lambda^*(t; 50) = 50(1 + \sin 2\pi t), \quad t > 0.$$

Specify the sequence $\{d_i, i \geq 1\}$ as follows:

- $d_i = 1$, if $1 \leq i \leq 100$;
- $d_{101} = 1.05d_{100}$, $d_{102} = 1.1d_{101}$, $d_{103} = 1.3d_{102}$, $d_{104} = 1.6d_{103}$, $d_{105} = 2d_{104}$;
- $d_i = 2.3^{i-105}d_{105}$, if $i \geq 106$.

Such sequence $\{d_i, i \geq 1\}$ guarantees that the assumptions of *Corollary 2* are fulfilled and one has bounds on the rate of convergence to the limiting characteristics given by (32) and (33) with $a = 1.7$, $R = 2$, $F = 100$.

In order to approximate the limiting characteristics for all three cases (i)–(iv), one can apply *Theorem 5* and *Theorem 8* from [31]. But firstly one has to specify the value i in the arrival intensity $\lambda^*(t; i)$, because as the arrival intensity (and thus load) grows the bigger state space is needed. Assume that $i \leq 50$ i.e. the maximum arrival intensity allowed is $\lambda^*(t; 50) = 50(1 + \sin 2\pi t)$. Then one can compute the solution of the forward Kolmogorov system (3) for the truncated process on the state space $\{0, 1, \dots, 155\}$ on the interval $[0, t^* + 1]$ with the initial condition $X(0) = 0$. Hence one finds the limiting idle probability and limiting mean value on the interval $[t^*, t^* + 1]$ with an error less than 10^{-4} , where $t^* = 5$ or $t^* = 7$.

Below one presents the plots of the limiting probability of the empty queue $p_0(t)$ and the limiting mean $\varphi(t)$ number of customer in the system for all each of the cases (i)–(iv).

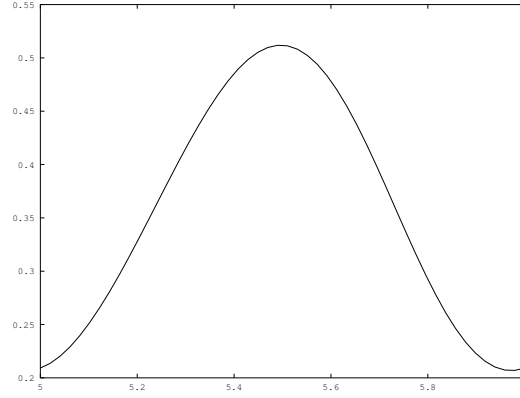


Figure 1: Case (i), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

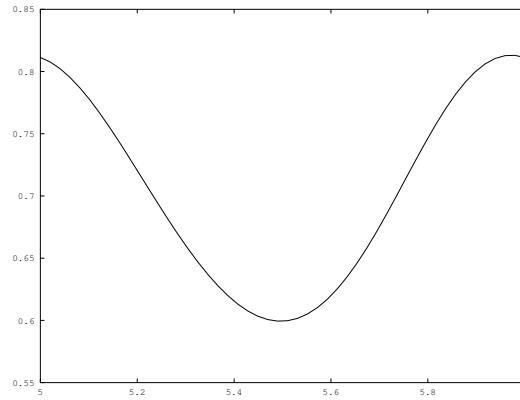


Figure 2: Case (i), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

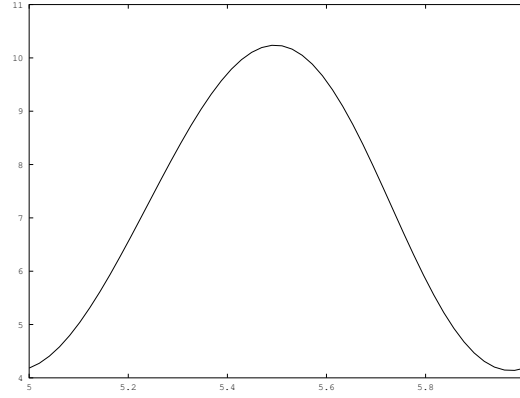


Figure 3: Case (i), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

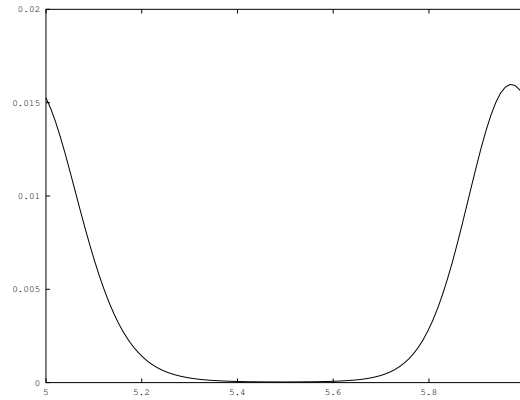


Figure 4: Case (i), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

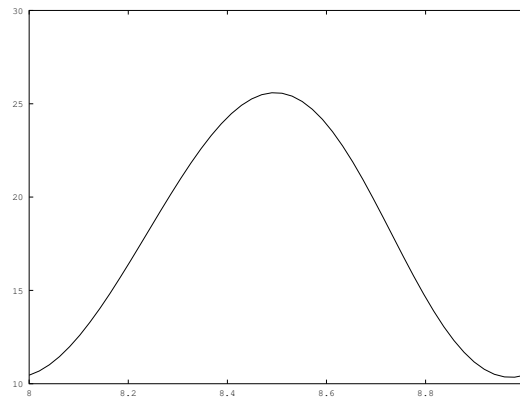


Figure 5: Case (i), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

Note from the figures above, that in case of high arrival intensity the limiting probability $p_0(t)$ of the empty queue here equals to 0 most of the time.

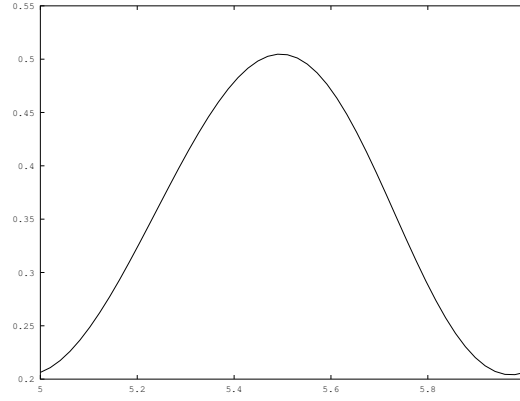


Figure 6: Case (ii), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

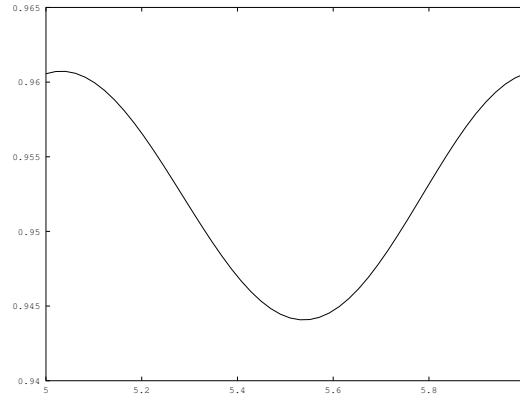


Figure 7: Case (ii), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

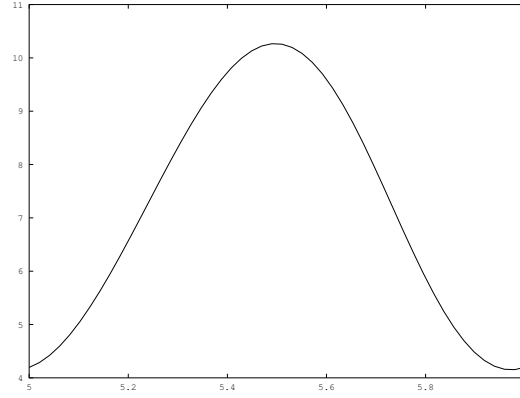


Figure 8: Case (ii), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

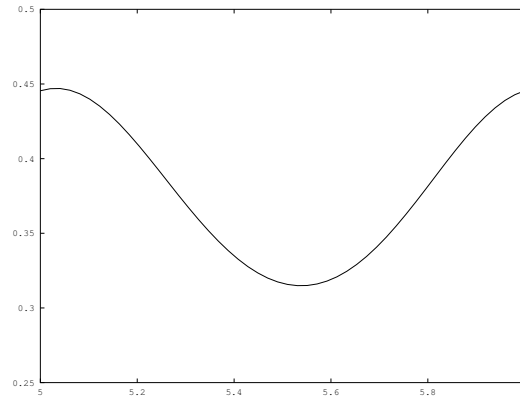


Figure 9: Case (ii), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

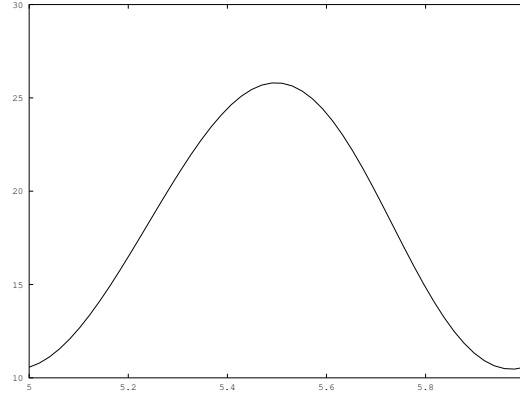


Figure 10: Case (ii), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

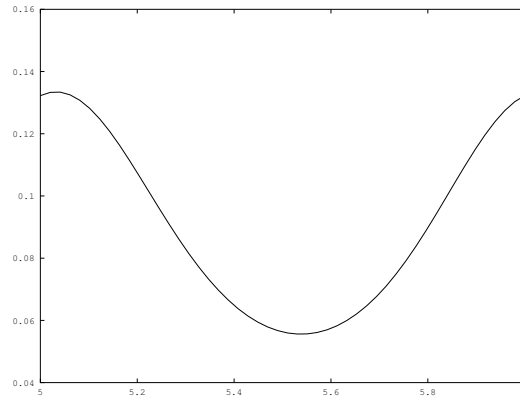


Figure 11: Case (ii), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

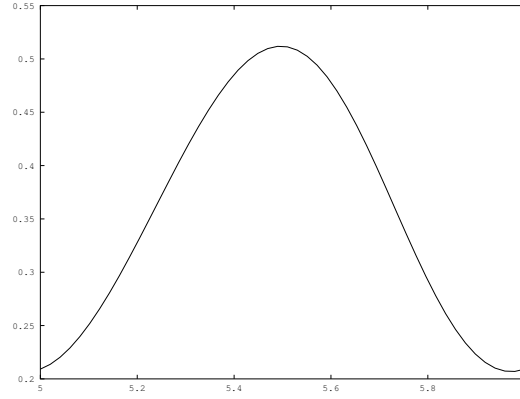


Figure 12: Case (iii), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

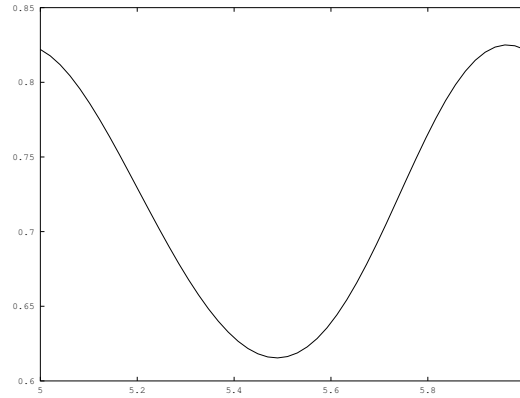


Figure 13: Case (iii), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

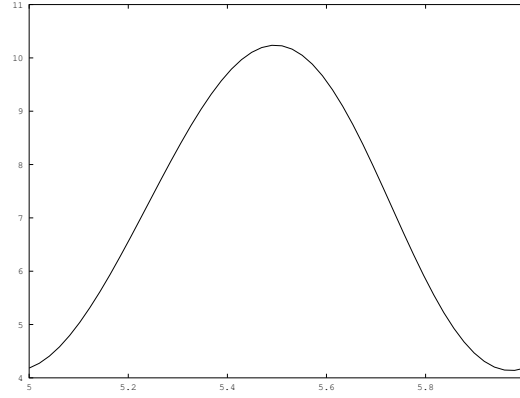


Figure 14: Case (iii), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

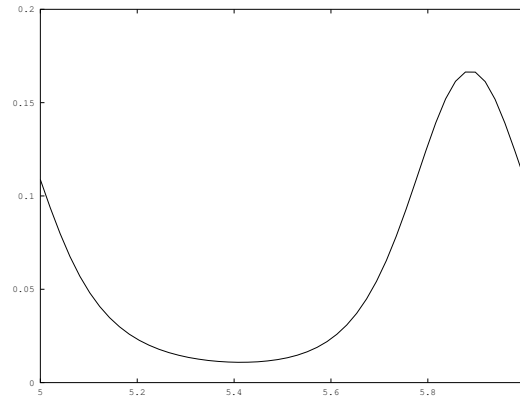


Figure 15: Case (iii), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

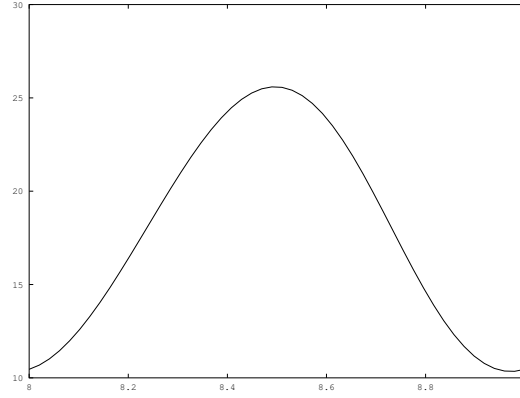


Figure 16: Case (iii), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [8, 9]$.

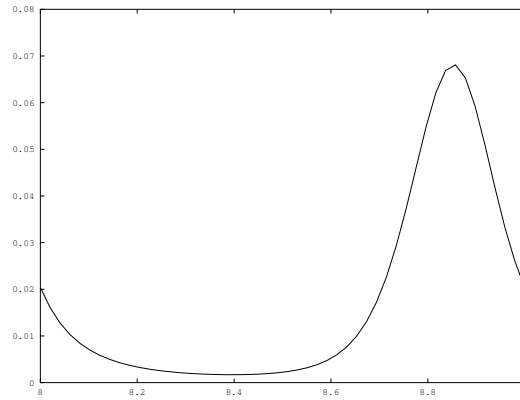


Figure 17: Case (iii), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [8, 9]$.

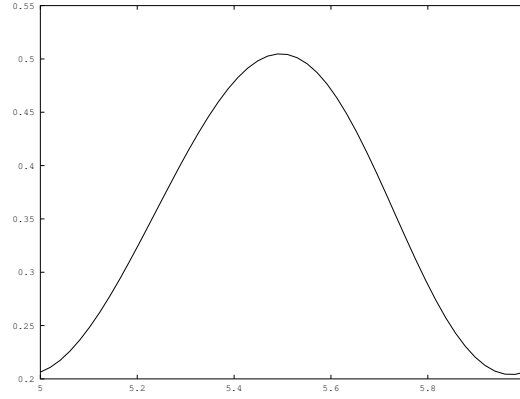


Figure 18: Case (iv), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

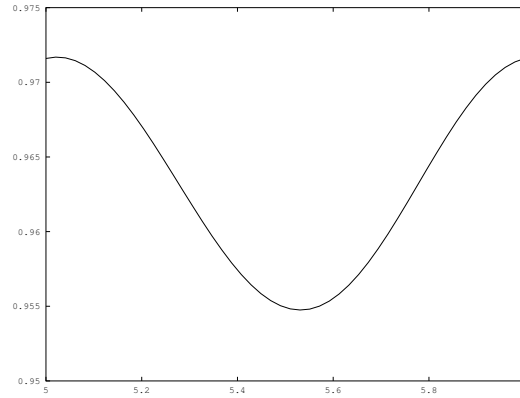


Figure 19: Case (iv), the arrival intensity is $\lambda^*(t; 10)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

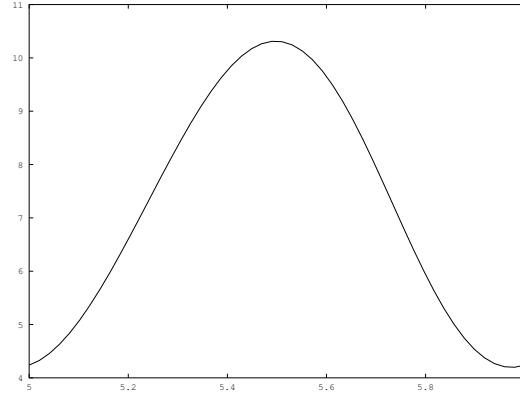


Figure 20: Case (iv), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

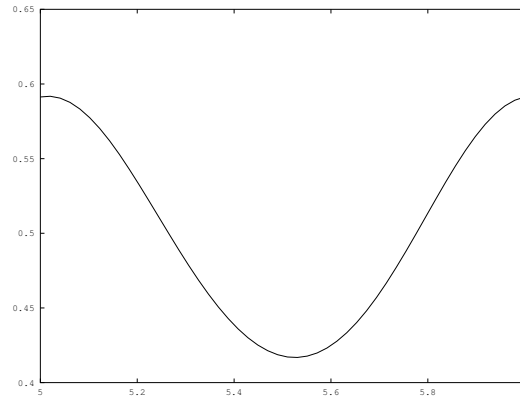


Figure 21: Case (iv), the arrival intensity is $\lambda^*(t; 20)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

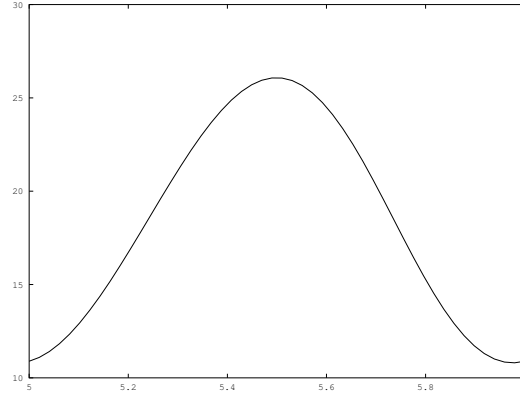


Figure 22: Case (iv), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting mean number $\varphi(t)$ of customers in the system for $t \in [5, 6]$.

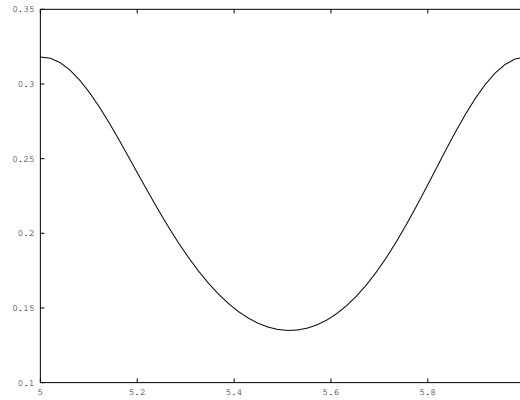


Figure 23: Case (iv), the arrival intensity is $\lambda^*(t; 50)$. Approximation of the limiting probability $p_0(t)$ of the empty queue for $t \in [5, 6]$.

6 Conclusion

From the presented figures one can see that the limiting mean number of customers in the system apparently does not depend on type of the system i.e. for all four different systems considered there is numerical evidence that the limiting means coincide. With respect to the probability of the empty queue one observes the clear dependence on the type of the system. These numerical evidences show one of the directions of further research: explanation these effects from the analytical point of view. Another direction is the generalization of the proposed method for other types of inhomogeneous queueing systems. One of the appealing candidates are queueing systems with balking, in which the arrival intensities decrease with the growth of the total number of customers in the system. Another one direction of research follows from [36] and is related to the optimization of no-wait probabilities in such queueing systems.

References

- [1] Daleckij, Ju.L., Krein, M.G.. Stability of solutions of differential equations in Banach space. Amer. Math. Soc. Transl. 43, 1974.
- [2] Di Crescenzo A., Giorno V., Nobile A.G., Ricciardi L.M. On the M/M/1 queue with catastrophes and its continuous approximation. *Queueing Syst.*, 2003, 43, 329–347.
- [3] Di Crescenzo A., Giorno V., Nobile A.G., Ricciardi L.M. A note on birth-death processes with catastrophes. *Statist. Probab. Lett.*, 2008, 78, 2248–2257.
- [4] Van Doorn E. A., Zeifman A. I., Panfilova T. L. Bounds and asymptotics for the rate of convergence of birth-death processes // Th. Prob. Appl., 2010, 54, 97–113.
- [5] Dudin A., Nishimura S. A BMAP/SM/1 queueing system with Markovian arrival input of disasters. *J. Appl. Probab.* 1999, 36, 868–881.
- [6] Dudin A., Karolik A. BMAP/SM/1 queue with Markovian input of disasters and non-instantaneous recovery. *Perform. Eval.* 2001, 45, 19–32.

- [7] Granovsky, B. L., & Zeifman, A. I. (2000). The N-limit of spectral gap of a class of birthdeath Markov chains. *Applied Stochastic Models in Business and Industry*, 16(4), 235-248.
- [8] Granovsky B. L., Zeifman A. I. Nonstationary Queues: Estimation of the Rate of Convergence // *Queueing Systems*, 2004, 46, p. 363–388.
- [9] Kartashov N. V. . Criteria for uniform ergodicity and strong stability of Markov chains with a common phase space, *Theory Probab. Appl.*, 30, 71-89. 1985.
- [10] Kartashov N. V. 1986. Strongly stable Markov chains, *Journal of Soviet Mathematics*, 34, 1493-149 (published in Russian in 1981).
- [11] Kartashov N. V. 1996. Strong stable Markov chains, Utrecht: VSP. Kiev: TBiMC.
- [12] Li, J., Zhang, L. Decay property of stopped Markovian bulk-arriving queues with c-servers. *Stochastic Models*, 1-13. 2016.
- [13] Li J., Anyue C. Decay property of stopped Markovian bulk-arriving queues. *Adv. in Appl. Probab.* 40 (2008), no. 1, 95–121.
- [14] Anyue C., Pollett P., Li J., Zhang H. Markovian bulk-arrival and bulk-service queues with state-dependent control. *Queueing Syst*, 2010, 64: 267–304.
- [15] Saloff-Coste L., Zuniga J. Convergence of some time inhomogeneous Markov chains via spectral techniques, *Stochastic Processes and their Applications*, Volume 117, Issue 8, August 2007, Pages 961-979
- [16] Schwarz J., Selinka G., Stollatz R. Performance analysis of time-dependent queueing systems: Survey and classification, *Omega*, Volume 63, September 2016, Pages 170-189,
- [17] Massey W. A., Whitt W. On analysis of the modified offered-load approximation for the nonstationary Erlang loss model // *Ann. Appl. Probab.*, 4, 1994, p. 1145–1160.
- [18] Meyn S.P., Tweedie R. Markov chains and stochastic stability. *Communications and Control Engineering Series*. Berlin: Springer-Verlag, 1993.

- [19] Mitrophanov A. Stability and exponential convergence of continuous-time Markov chains // J. Appl. Probab., 40, 2003, p. 970–979.
- [20] Nelson, R., Towsley, D., & Tantawi, A. N. (1988). Performance analysis of parallel processing systems. IEEE Transactions on software engineering, 14(4), 532-540.
- [21] Satin Ya. A., Zeifman A., Korotysheva A. On the Rate of Convergence and Truncations for a Class of Markovian Queueing Systems // Theory Probab. Appl., 57(3), 2013. p.529-539.
- [22] Zeifman A. I. Stability for continuous-time nonhomogeneous Markov chains // Lect. Notes Math., 1155, 1985, p. 401–414.
- [23] Zeifman A. I. Quasi-ergodicity for nonhomogeneous continuous-time Markov chains // J. Appl. Probab., 26, 1989, p. 643–648.
- [24] Zeifman A. I. Some estimates of the rate of convergence for birth and death processes // J. Appl. Probab. 28, 1991, p. 268–277.
- [25] Zeifman A. I. On the estimation of probabilities for birth and death processes // J. Appl. Probab., 32, 1995, p. 623–634.
- [26] Zeifman A. I. Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes // Stoch. Proc. Appl., 59, 1995, p. 157–173.
- [27] Zeifman A., Leorato S., Orsingher E., Satin Ya., Shilova G. Some universal limits for nonhomogeneous birth and death processes // Queueing systems, 52, 2006, p. 139–151.
- [28] Zeifman A., Korotysheva A. Perturbation Bounds for Mt/Mt/N Queue with Catastrophes // Stochastic Models, 28:1, 2012. p. 49-62.
- [29] Zeifman, A., Satin, Y., Panfilova, T. (2013). Limiting characteristics for finite birthdeath-catastrophe processes. Mathematical biosciences, 245(1), 96-102.
- [30] Zeifman A., Korotysheva A., Korolev V. , Satin Y. , Bening V. Perturbation bounds and truncations for a class of Markovian queues // Queueing Systems, 76, 2014. 205–221.

- [31] Zeifman A. , Satin Ya., Korolev V., Shorgin S. On truncations for weakly ergodic inhomogeneous birth and death processes // International Journal of Applied Mathematics and Computer Science, 2014, 24, 503–518.
- [32] Zeifman, A. I., Korolev, V. Y. (2014). On perturbation bounds for continuous-time Markov chains. *Statistics & Probability Letters*, 88, 66–72.
- [33] Zeifman A. , Satin Ya., Korotysheva A., Korolev V., Shorgin S., Razumchik R. Ergodicity and perturbation bounds for inhomogeneous birth and death processes with additional transitions from and to origin // *Int. J. Appl. Math. Comput. Sci*, 2015, 25(4), 503–518.
- [34] Zeifman, A. I., Korolev, V. Y. (2015). Two-sided bounds on the rate of convergence for continuous-time finite inhomogeneous Markov chains. *Statistics & Probability Letters*, 103, 30-36.
- [35] Zeifman A., Korotysheva A., Korolev V., Satin Ya. Truncation bounds for approximations of inhomogeneous continuous-time Markov chains // *Th. Prob. Appl*, 2016, submitted. 2016.
- [36] Zeifman A., Korotysheva A., Satin Ya., Shilova G. Razumchik R., Korolev V., Shorgin S. Uniform In Time Bounds For No-Wait Probability In Queues Of Mt/Mt/S Type, *ECMS 2016 Proceedings* edited by: Thorsten Claus, Frank Herrmann, Michael Manitz, Oliver Rose European Council for Modeling and Simulation. P. 676-684.